DERIVED INTERSECTIONS IN QUASI-SMOOTH AFFINE SCHEMES

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Abstract. In this paper, we investigate derived intersections in affine schemes. In particular, we study when derived intersections in quasi-smooth affine schemes are quasi-isomorphic to symmetric algebras of chain complexes, and prove two cases when this is always possible. We also explore the connection between derived intersections and the cotangent complex.

Contents

1. Introduction

In what follows, all rings will be commutative with 1. Given a ring R and ideals I and J, $X = \text{Spec } R/I$ and $Y = \text{Spec } R/J$ are closed subschemes of $S = \text{Spec } R$. Their *scheme-theoretic intersection* is the fiber product in the category of schemes $X \times_S Y = \text{Spec}(R/I \otimes_R R/J)$. The scheme-theoretic intersection captures intersection multiplicity information, while the set-theoretic intersection does not. For example, if we take the intersection of the parabola $y - x^2 = 0$ with the line $y = 0$, we get Spec $k[x]/x^2$ which is a ring of length two. This corresponds to the fact that in some sense the intersection is of multiplicity 2, since the parabola is tangent to the line.

However, in certain contexts, the scheme-theoretic intersection does not contain enough information to determine the correct intersection multiplicity; this can occur when the intersection has the "wrong dimension." Instead of an ordinary tensor product $R/I \otimes_R R/J$ we need to consider the information contained in the *derived tensor product* $R/I \otimes^L_R R/J$. The derived tensor product is not just a ring but a *differential-graded ring*; that is, it also has the structure of a chain complex. In general, [Ser00] defined the intersection multiplicity at a point p to be the Euler characteristic of the derived tensor product of the local rings at p, which can be computed as an alternating sum of lengths of Tor groups:

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$$\chi(\mathcal{O}_{X,p} \otimes^L_{\mathcal{O}_{S,p}} \mathcal{O}_{Y,p}) = \sum_{i=0}^{\infty} (-1)^i l(\mathrm{Tor}(\mathcal{O}_{X,p}, \mathcal{O}_{Y,p}).$$

In the setting of derived algebraic geometry, we define new kinds of geometric objects to capture information about this sort of situation. Rather than considering the basic objects of algebraic geometry to be schemes, i.e. locally built out of the spectra of commutative rings, we generalize to differential graded schemes, which are locally built out of the "derived spectra" of differential graded rings. For more on the motivation for derived algebraic geometry beyond just the context of intersection theory, see the introduction to [Lur18].

In this paper, we collect and prove some results on derived intersections of affine schemes. Specifically, we investigate the question of when the derived intersection can be written as a symmetric algebra. Our questions are motivated by results by Arinkin, Căldăraru, and Hablicsek [ACH14], who studied formality of derived intersections. However, they treat the global case and only consider smooth schemes, whereas we only consider the local case, but also consider singular schemes. We also draw inspiration from a paper by Arinkin and Căldăraru [AC12], who investigated when a derived self-intersection of a closed subscheme inside a smooth scheme is a fibration over the subscheme. In general, there is a morphism of spaces from the underived intersection to the derived intersection, and formality refers to cases where the corresponding ring homomorphism makes the structure sheaf of the derived intersection a symmetric algebra (in the differential-graded sense, as we will describe in the next section) over the structure sheaf of the underived intersection. This should remind the reader of algebro-geometric vector bundles, and we can describe the derived intersection as a "shifted vector bundle" over the underived intersection.

Furthermore, our work is motivated by actions of the multiplicative group $\mathbb{G}_m$. We have that a $\mathbb{G}_m$-action on an affine scheme X is equivalent to a grading on X. Moreover, we say that a $\mathbb{G}_m$-action on X is *contracting* if the grading induced on X is in degrees ≤ 0 (using cohomological conventions). Thus, if a derived intersection can be written non-trivially as a symmetric algebra, then there exists a contracting $\mathbb{G}_m$-action on the derived intersection, since the symmetric algebra is graded.

The bulk of this paper is devoted to exposition of the basic notions involved in the story. We begin by presenting a gentle introduction to differential graded algebras, and then describe Koszul-Tate resolutions. These allow us to resolve an algebra by a differential graded algebra, and are our main computational tool. We then compute many examples of derived intersections. Everything we do will be local (affine) and thus completely algebraic.

After this, we take a detour and provide a brief exposition of Kähler differentials and the cotangent complex. We then present some of our results on derived intersections in quasi-smooth schemes. The main results are the following: let R be a regular local ring over base field k, and let A, I, and J be ideals of R generated by regular sequences such that $A \subset I$, $A \subset J$, and R/I and R/J are regular local rings. Let $\overline{R} = R/A$, $\overline{I} = I/A$, and $\overline{J} = J/A$. Then the following theorems hold:

Theorem 1. *If $I \subset J$, then $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\overline{R}/\overline{J}} \mathbb{L}_{Y/k}$ where $Y = Spec\ R/J$.*

Theorem 2. *Let $A = (a_1, \ldots a_r)$ and $I = (f_1, \ldots, f_n)$. If $A \subset IJ$, then $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\widetilde{R}} \widetilde{R}^r[2]$, where $\widetilde{R}$ is the dg algebra $\overline{R}/\overline{J}[e_1, \ldots, e_n]$ with $|e_i| = -1$ and with differential $d(e_i) = f_i$.*

Subsequently, we present some computational results that explicitly describe the cohomology of certain derived self-intersections. Finally, given a quasi-smooth affine scheme X and an automorphism f of X, we state a question about the derived intersection of the diagonal of X with the graph of f.

2. Preliminaries on dg algebras and dg modules

Throughout this paper, k will be a field of characteristic zero. As budding algebraic geometers, we will adopt *cohomological* indexing conventions.

Definition 3. A *differential graded algebra (dg algebra) over k* is a nonpositively graded cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that

$$d_{n+m}(ab) = d_n(a)b + (-1)^n a d_m(b)$$

and such that $\bigoplus A^n$ becomes an associative and unital R-algebra. If $a \in A^n$, we say $|a| = n$. We will denote the differential as simply d.

We often just write (A, d) for $A^\bullet$ and think of this as an associative unital R-algebra endowed with a $\mathbb{Z}_{\leq 0}$-grading and an R-linear operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say "let (A, d) be a differential graded algebra".

Definition 4. A *homomorphism of dg algebras* $f : (A, d) \to (B, d)$ is an algebra map $f : A \to B$ compatible with the gradings and the differential d.

A *quasi-isomorphism* of dg algebras is a homomorphism of dg algebras that induces isomorphisms on all cohomology groups.

Definition 5. A differential graded algebra (A, d) is *skew-commutative* if $ab = (-1)^{|a||b|}ba$. for homogeneous elements a and b. Note that since k is of characteristic zero, this implies $a^2 = 0$ when $|a|$ is odd.

Remark 6. Henceforth we shall use "dg algebra" to mean skew-commutative differential graded algebra. These are the basic objects of study in this paper. Skew-commutative dg algebras are often called "cdgas" in the literature.

It is possible to describe dg algebras using generators and relations, just as with ordinary algebras. However, one must watch out for the skew-commutativity in the sign rule. In the first few following examples, the differential will be trivial, to focus on the consequences of grading and skew-commutativity.

Example 7. We will describe a "polynomial dg algebra" in one variable, which is homogenous in degree 1, with trivial differential. Let $A = k[e_1]$, $|e_1| = -1$, and $d(e_1) = 0$. Thus A_n is generated as a vector space by e_1^n. However, $(e_1)^2 = 0$, so $A_2 = A_3 = \dots$ are trivial. Thus A looks like

$$\cdots \to 0 \to k(e_1) \to k$$

where the differentials are zero.

Example 8. Similarly, let $A = k[e_1, \dots e_n]$, $|e_i| = -1$, $d(e_i) = 0$. The sign rule dictates that $e_ie_j = -e_je_i$ when $i \neq j$ and $e_i^2 = 0$. Thus A is an exterior algebra on the generators $e_1, \dots e_n$, and looks like

$$\cdots \to 0 \to 0 \to \wedge^n(ke_1 \oplus \cdots \oplus ke_n) \to \cdots \to \wedge^2(ke_1 \oplus \cdots \oplus ke_n) \to ke_1 \oplus \cdots \oplus ke_n \to k$$

where the differentials are again zero. The complex is entirely concentrated in degrees zero through $-n$.

In the world of differential graded algebras, skew-commutativity substitutes for commutativity. Although we write A as $k[e_1, \dots e_n]$, A is in fact an exterior algebra, which is anti-commutative! *However, we will use the terms "polynomial algebra" and "symmetric algebra" to describe such skew-commutative algebras.* By analogy with symmetric algebras in classical commutative algebra, we can alternatively describe the algebra of the previous example as $\operatorname{Sym}_k k^n[1]$. Here $k^n[1]$ refers to the complex which is k^n in degree negative one and zero everywhere else.

Example 9. We will now consider polynomial variables in even degree. Let $A = k[e_1, \dots e_n]$, $|e_i| = -2$, $d(e_i) = 0$. The sign rule dictates that $e_ie_j = e_je_i$, so what we have is a commutative algebra. Thus A looks like

$$\cdots \to \operatorname{Sym}^{n+1}(ke_1 \oplus \cdots \oplus ke_n) \to 0 \to \operatorname{Sym}^n(ke_1 \oplus \cdots \oplus ke_n) \to \cdots \to 0 \to ke_1 \oplus \cdots \oplus ke_n \to 0 \to k$$

where the differentials are again zero. Notice that the complex is concentrated in even degrees. Also notice that unlike the previous example, the complex is not bounded.

Remark 10. Again, we can describe the previous algebra as $\operatorname{Sym}_k k^n[2]$. This algebra behaves much more like an honest symmetric algebra; it is actually commutative. As the previous two examples indicate, dg algebras simultaneously generalize classical symmetric algebras and exterior algebras.

Example 11. We will now consider an example where the differential is nontrivial. Let $A = k[e_1, \varepsilon_1]$, $|e_1| = -1$, $|\varepsilon_1| = -2$, $d(e_i) = 0$, $d(\varepsilon_1) = e_1$. Thus $e_1^2 = 0$, and $\varepsilon_1^n e_1 = (-1)^{2\times 1} e_1 \varepsilon_1^n = e_1 \varepsilon_1^n$. So there is only one generator of each degree, $\varepsilon_1^{i/2}$ if i is even and $\varepsilon^{(i-1)/2}e_1$ is i is odd.

Now, we claim $d(\varepsilon_1^n) = n\varepsilon_1^{n-1}e_1$. We can prove this inductively: it is true for $n = 1$, and $d(\varepsilon_1^n) = d(\varepsilon_1^{n-1})\varepsilon_1 + \varepsilon_1^{n-1}d(\varepsilon_1) = (n-1)\varepsilon_1^{n-2}\varepsilon_1 + \varepsilon_1^{n-1}e_1 = n\varepsilon_1^{n-1}e_1$. Next, $d(\varepsilon_n^n e_1) = d(\varepsilon_n^n)e_1 + \varepsilon_n^n d(e_1) = n\varepsilon_1^{n-1}e_1^2 = 0$. So our complex looks like

$$\dots \xrightarrow{3} k\varepsilon_1^2 e_1 \xrightarrow{0} k\varepsilon_1^2 \xrightarrow{2} k\varepsilon_1 e_1 \xrightarrow{0} k\varepsilon_1 \xrightarrow{1} ke_1 \xrightarrow{0} 0.$$

Definition 12. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The *tensor product* of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$$

endowed with differential d defined by the rule $d(a \otimes b) = d(a) \otimes b + (-1)^{|a|} a \otimes d(b)$ where $m = \deg(a)$.

Remark 13. Observe that it is easy to take tensor products of polynomial dg algebras. For example, $k[e_1] \otimes k[e_2] \cong k[e_1, e_2]$, and the differential can be computed from the rule.

Now we will discuss dg modules.

Definition 14. Let (A, d) be a (not necessarily skew-commutative) differential graded algebra over k. A (right) *differential graded module* M over A is a right A-module M which has a grading $M = \bigoplus M^n$ and a differential d such that $M^n A^m \subset M^{n+m}$, such that $d(M^n) \subset M^{n+1}$, and such that

$$d(ma) = d(m)a + (-1)^n m d(a)$$

for $a \in A$ and $m \in M^n$.

Definition 15. A *homomorphism of differential graded modules* $f : M \to N$ is an A-module map compatible with gradings and differentials. The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,d)}$.

We can similarly define *left differential graded A-modules* and *differential graded A-bimodules*. When A is strictly commutative, they are the same we just call them *differential graded A-modules*.

Definition 16. Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. For any $k \in \mathbb{Z}$ we define the *k-shifted module* $M[k]$ as follows:

(1) $M[k] = M$ as A-modules ,
(2) $M[k]^n = M^{n+k}$,
(3) $d_{M[k]} = (-1)^k d_M$.

For a morphism $f : M \to N$ of differential graded A-modules we let $f[k] : M[k] \to N[k]$ be the map equal to f on underlying A-modules. This defines a functor $[k] : \mathrm{Mod}_{(A,d)} \to \mathrm{Mod}_{(A,d)}$.

For more details on dg algebras and dg modules, see [Man] or the Stacks Project [Sta18, Tag 061U].

3. Application to Koszul and Tate resolutions

The formalism of dg algebras allows us to give a fairly elegant treatment of the Koszul complex. The Koszul complex is a homological tool that allows to study regular sequences.

Definition 17. A sequence of elements $f_1, \dots f_n \in R$ is a *regular sequence* if $(f_1, \dots f_n) \neq R$ and f_i is a nonzerodivisor in $R/(f_1, \dots f_{i-1})$, for all i. In this case $I = (f_1, \dots f_n)$ is called a *regular ideal*.

Geometrically, regular sequences correspond to (local) complete intersections. The Koszul complex is useful because of the following theorem:

Theorem 18. *If $f_1, \dots f_n$ is a regular sequence, then $H_i(K(f_1, \dots f_n)) = 0$ for $i > 0$.*

Proof. See [Eis95, Corollary 17.5]. [1] □

We may define the Koszul complex as follows:

Definition 19. The *Koszul complex* $K(R; f_1, \dots f_n)$ for a sequence of elements $f_i \in R$ is the polynomial dg algebra $R[e_1, \dots e_n]$, where $|e_i| = -1$ and $d(e_i) = f_i$ for all i.

Theorem 20. *If $f_1, \dots f_n$ is a regular sequence, then the Koszul complex is a resolution of $R/(f_1, \dots f_n)$*

Proof. Clearly $H_0(K(R; f_1, \dots f_n)) = R/(f_1, \dots f_n)$. Exactness everywhere else follows from Theorem 18. □

[1] Eisenbud uses the dual Koszul complex; one must translate $H^k(K(*)) \cong H_{n-k}(K(*))$.

In the above, we resolved an algebra by a polynomial dg algebra. In general, it is possible to resolve any quotient algebra R/I in such a way. The way this works is, first, add polynomial variables in degree one, which are sent by the differential to generators of I. Then, we add variables in degree two, which are sent to nontrivial cycles in degree one. Then we add variables in degree three, which are sent to nontrivial cycles in degree two. And so on. Such a resolution is called a *Koszul-Tate resolution*, and the concept was originally introduced in a 1957 paper by Tate [Tat57]. We shall heavily make use the special case with only degree one and degree two polynomial variables, as follows:

Theorem 21. *Let $A = (a_1, \dots a_r)$ and $I = (f_1, \dots f_n)$ be regular ideals such that $A \subset I$. Write $a_j = \sum_{i=1}^{n} c_{ji} f_i$ where $c_{ji} \in R$. Let $\overline{R} = R/A$ and $\overline{I} = I/A$. Then the polynomial dg algebra*

$$\overline{R}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r],$$

where $|e_i| = -1$, $|\varepsilon_j| = -2$, $d(e_i) = f_i$ and $d(\varepsilon_j) = \sum_{i=1}^{n} c_{ji} e_i$ is a resolution of $\overline{R}/\overline{I}$ as an $\overline{R}$-module.

Proof. [Tat57, Theorem 4]. □

4. Computations of derived intersections in quasi-smooth affine schemes

In this section, we present some examples of derived intersections. We begin by introducing the general setting that will be used throughout this paper. We will focus on quasi-smooth affine schemes.

Definition 22. A dg algebra A is *quasi-smooth* if for every prime ideal p of A, we have that A_p is quasi-isomorphic (as a chain complex) to the Koszul complex $K(B; f_1, \dots, f_r)$ for some smooth classical ring B.

For affine schemes, being quasi-smooth is equivalent to being a locally complete intersection. An affine scheme is a locally complete intersection if it is the spectrum of a local complete intersection ring, that is, a Noetherian local ring whose completion is the quotient of a regular local ring by an ideal generated by a regular sequence.

Unless otherwise specified, our setting for computing derived intersections is as follows: S will be a quasi-smooth affine scheme, while X and Y will be closed subschemes of X. We want to compute the derived intersection

$$W = X \times_S^L Y.$$

From the algebraic point of view, let us consider the following setup. R is a regular local ring over a field k, and A, I and J are ideals of R generated by regular sequences such that $A \subset I$ and $A \subset J$. Furthermore, we require R/I and R/J to be regular local rings. Let $\overline{R} = R/A$, $\overline{I} = I/A$, and $\overline{J} = J/A$. Then we can set $X = \text{Spec } \overline{R}/\overline{I}$, $Y = \text{Spec } \overline{R}/\overline{J}$, and $S = \text{Spec } \overline{R}$. Note that $\overline{R}$ is a local complete intersection ring. We are interested in the derived tensor product of quotient rings over $\overline{R}$

$$\overline{R}/\overline{I} \otimes_{\overline{R}}^{L} \overline{R}/\overline{J}.$$

We will use the Tate resolution stated in Theorem 21 to calculate the derived tensor product in quasi-smooth affine schemes.

It is important to note that the derived intersection is only well-defined up to quasi-isomorphism.

Remark 23. In our examples, all coordinate rings and ideals are actually localized at the origin since we are working with regular local rings. For convenience, we omit the localization notation in this section, and all the computations are still correct. For example $R = k[x, y]$ means $R = k[x, y]_{(x,y)}$ in this section. Also, k will always be a field of characteristic 0.

Remark 24. Even though $R/I \cong \overline{R}/\overline{I}$ by the third isomorphism theorem of rings, we will use $\overline{R}/\overline{I}$ as our notation for clarity of context.

4.1. Computations of derived self-intersections.

In this subsection, we will focus on self-intersections of quasi-smooth affine schemes, i.e., the case where $I = J$, similar to [AC12].

Example 25. Let $R = k[x, y]$, $A = (xy)$, and $I = (x)$. The Tate resolution of $\overline{R}/\overline{I}$ is the polynomial dg algebra $\overline{R}[e, \varepsilon]$, where $|e| = -1$, $|\varepsilon| = -2$, $d(e) = x$, and $d(\varepsilon) = ye$. Thus, we obtain that

$$\overline{R}/\overline{I} \otimes_{\overline{R}}^{L} \overline{R}/\overline{I} \cong \overline{R}[e, \varepsilon] \otimes_{\overline{R}} \overline{R}/\overline{I} \cong \overline{R}/\overline{I}[e, \varepsilon].$$

Note $\overline{R}/\overline{I} \cong k[y]$, so $d(e) = 0$ and $d(\varepsilon) = ye$ in $\overline{R}/\overline{I}$. We may write $\overline{R}/\overline{I}[e,\varepsilon]$ as

$$\cdots \xrightarrow{0} \overline{R}/\overline{I}\varepsilon^2 \xrightarrow{d} \overline{R}/\overline{I}e\varepsilon \xrightarrow{0} \overline{R}/\overline{I}\varepsilon \xrightarrow{d} \overline{R}/\overline{I}e \xrightarrow{0} \overline{R}/\overline{I}.$$

Since

$$d(\varepsilon^n) = (d\varepsilon)\varepsilon^{n-1} + \varepsilon(d\varepsilon^{n-1}) = nye\varepsilon^{n-1},$$

and

$$d(e\varepsilon^n) = (de)\varepsilon^n - e(d\varepsilon^n) = 0,$$

we could rewrite the complex as

$$\cdots \xrightarrow{0} \overline{R}/\overline{I} \xrightarrow{\cdot 2y} \overline{R}/\overline{I} \xrightarrow{0} \overline{R}/\overline{I} \xrightarrow{\cdot y} \overline{R}/\overline{I} \xrightarrow{0} \overline{R}/\overline{I}.$$

In other words,

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{I} \cong \mathrm{Sym}_{k[y]}(k[y] \xrightarrow{\cdot y} k[y]).$$

Example 26. Let $R = k[x,y]$, $A = (xy)$, and $I = (x,y)$. The Tate resolution of $\overline{R}/\overline{I}$ is the polynomial dg algebra $\overline{R}[e_1,e_2,\varepsilon]$, where $|e_1| = -1$, $|e_2| = -1$, $|\varepsilon| = -2$, $d(e_1) = x$, $d(e_2) = y$, and $d(\varepsilon) = ye_1$. Thus, we obtain that

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{I} \cong \overline{R}[e_1,e_2,\varepsilon] \otimes_{\overline{R}} \overline{R}/\overline{I} \cong \overline{R}/\overline{I}[e_1,e_2,\varepsilon].$$

Note $\overline{R}/\overline{I} \cong k$, so $d(e_1) = 0$, $d(e_2) = 0$, and $d(\varepsilon) = 0$ in $\overline{R}/\overline{I}$. Therefore the derived tensor product $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{I}$ is quasi-isomorphic to the chain complex

$$\cdots \xrightarrow{0} \overline{R}/\overline{I}\varepsilon^2 \oplus \overline{R}/\overline{I}e_1e_2\varepsilon \xrightarrow{0} \overline{R}/\overline{I}e_1\varepsilon \oplus \overline{R}/\overline{I}e_2\varepsilon \xrightarrow{0} \overline{R}/\overline{I}\varepsilon \oplus \overline{R}/\overline{I}e_1e_2 \xrightarrow{0} \overline{R}/\overline{I}e_1 \oplus \overline{R}/\overline{I}e_2 \xrightarrow{0} \overline{R}/\overline{I}.$$

In other words,

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{I} \cong \mathrm{Sym}_k(k^2[1] \oplus k[2]).$$

4.2. **Computations of general derived intersections.** In this subsection, we will focus on general derived intersections. Note that resolving $\overline{R}/\overline{I}$ first and then tensor with $\overline{R}/\overline{J}$ will give us the same (up to quasi-isomorphism) result as resolving $\overline{R}/\overline{J}$ first and then tensoring with $\overline{R}/\overline{I}$.

Example 27. Let $R = k[x,y]$, $A = (xy)$, $I = (x)$, and $J = (y)$. From example 19, the Tate resolution of $\overline{R}/\overline{I}$ is the polynomial dg algebra $\overline{R}[e,\varepsilon]$, where $|e| = -1$, $|\varepsilon| = -2$, $d(e) = x$, and $d(\varepsilon) = ye$. Thus, we obtain that

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}[e,\varepsilon] \otimes_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}/\overline{J}[e,\varepsilon].$$

Note $\overline{R}/\overline{J} \cong k[x]$, so $d(e) = x$ and $d(\varepsilon) = 0$ in $\overline{R}/\overline{J}$. We may write $\overline{R}/\overline{J}[e,\varepsilon]$ as

$$\cdots \xrightarrow{d} \overline{R}/\overline{J}\varepsilon^2 \xrightarrow{0} \overline{R}/\overline{J}e\varepsilon \xrightarrow{d} \overline{R}/\overline{J}\varepsilon \xrightarrow{0} \overline{R}/\overline{J}e \xrightarrow{d} \overline{R}/\overline{J}.$$

Since

$$d(\varepsilon^n) = (d\varepsilon)\varepsilon^{n-1} + \varepsilon(d\varepsilon^{n-1}) = 0,$$

and

$$d(e\varepsilon^n) = (de)\varepsilon^n - e(d\varepsilon^n) = x\varepsilon^n,$$

we could rewrite the complex as

$$\cdots \xrightarrow{\cdot x} \overline{R}/\overline{J} \xrightarrow{0} \overline{R}/\overline{J} \xrightarrow{\cdot x} \overline{R}/\overline{J} \xrightarrow{0} \overline{R}/\overline{J} \xrightarrow{\cdot x} \overline{R}/\overline{J}.$$

In other words,

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{k[x]}(k[x] \xrightarrow{\cdot x} k[x]).$$

Example 28. Let $R = k[x,y]$, $A = (y(y-x^2-1))$, $I = (y-x^2-1)$, and $J = (y)$. The Tate resolution of $\overline{R}/\overline{I}$ is the polynomial dg algebra $\overline{R}[e,\varepsilon]$, where $|e| = -1$, $|\varepsilon| = -2$, $d(e) = y - x^2 - 1$ and $d(\varepsilon) = ye$. Thus, we obtain that

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}[e,\varepsilon] \otimes_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}/\overline{J}[e,\varepsilon].$$

Note $\overline{R}/\overline{J} \cong k[x]$, so $d(e) = x^2 + 1$ and $d(\varepsilon) = 0$ in $\overline{R}/\overline{J}$. We may write $\overline{R}/\overline{J}[e,\varepsilon]$ as

$$\cdots \xrightarrow{d} \overline{R}/\overline{J}\varepsilon^2 \xrightarrow{0} \overline{R}/\overline{J}e\varepsilon \xrightarrow{d} \overline{R}/\overline{J}\varepsilon \xrightarrow{0} \overline{R}/\overline{J}e \xrightarrow{d} \overline{R}/\overline{J}.$$

Since

$$d(\varepsilon^n) = (d\varepsilon)\varepsilon^{n-1} + \varepsilon(d\varepsilon^{n-1}) = 0,$$

and

$$d(e\varepsilon^n) = (de)\varepsilon^n - e(d\varepsilon^n) = (x^2+1)\varepsilon^n,$$

we could rewrite the complex as

$$\cdots \xrightarrow{\cdot(x^2+1)} \overline{R}/\overline{J} \xrightarrow{0} \overline{R}/\overline{J} \xrightarrow{\cdot(x^2+1)} \overline{R}/\overline{J} \xrightarrow{0} \overline{R}/\overline{J} \xrightarrow{\cdot(x^2+1)} \overline{R}/\overline{J}.$$

In other words,

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{k[x]}(k[x] \xrightarrow{\cdot(x^2+1)} k[x]).$$

Example 29. Let $R = k[x,y,z]$, $A = (xyz)$, $I = (xy)$, and $J = (xz)$. The Tate resolution of $\overline{R}/\overline{I}$ is the polynomial dg algebra $\overline{R}[e,\varepsilon]$, where $|e| = -1$, $|\varepsilon| = -2$, $d(e) = xy$ and $d(\varepsilon) = ze$. Thus, we obtain that

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}[e,\varepsilon] \otimes_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}/\overline{J}[e,\varepsilon].$$

We may write $\overline{R}/\overline{J}[e,\varepsilon]$ as

$$\cdots \xrightarrow{d} \overline{R}/\overline{J}\varepsilon^2 \xrightarrow{d} \overline{R}/\overline{J}e\varepsilon \xrightarrow{d} \overline{R}/\overline{J}\varepsilon \xrightarrow{d} \overline{R}/\overline{J}e \xrightarrow{d} \overline{R}/\overline{J}.$$

Since

$$d(\varepsilon^n) = (d\varepsilon)\varepsilon^{n-1} + \varepsilon(d\varepsilon^{n-1}) = nze\varepsilon^{n-1},$$

and

$$d(e\varepsilon^n) = (de)\varepsilon^n - e(d\varepsilon^n) = xy\varepsilon^n,$$

we could rewrite the complex as

$$\cdots \xrightarrow{\cdot xy} \overline{R}/\overline{J} \xrightarrow{\cdot 2z} \overline{R}/\overline{J} \xrightarrow{\cdot xy} \overline{R}/\overline{J} \xrightarrow{\cdot z} \overline{R}/\overline{J} \xrightarrow{\cdot xy} \overline{R}/\overline{J}.$$

5. Kähler differentials and the cotangent complex

5.1. **Kähler differentials.** Given a k-algebra R, the module of of Kähler differentials $\Omega_{R/k}$ corresponds to the cotangent sheaf on Spec R. Its elements can be interpreted as "infinitesimal transformations", and in particular, elements of the dual of $\Omega_{R/k}$ can profitably be interpreted as vectors fields on the variety Spec R.

Definition 30. Let R be a k-algebra. We define the *module of Kähler differentials* of R over k, written $\Omega_{R/k}$, to be the R-module generated by the set $\{d(f) \mid f \in R\}$ subject to the relations

$$d(r_1r_2) = r_1d(r_2) + r_2d(r_1) \qquad \text{(Leibniz rule)}$$

$$d(c_1r_1 + c_2r_2) = c_1d(r_1) + c_2d(r_2) \qquad (k\text{-linearity})$$

for all $r_1, r_2 \in R$ and $c_1, c_2 \in k$.

We now list several useful properties of Kähler differentials, the proofs of which can be found in [Eis95, Chapter 16].

Given an R-module M, we let $\mathrm{Der}_k(R,M)$ be the set of all k-linear maps $R \to M$ satisfying the Leibniz rule. Elements of $\mathrm{Der}_k(R,M)$ are called k-linear derivations. We then have the following isomorphism:

$$\mathrm{Der}_k(R,M) \cong \mathrm{Hom}_R(\Omega_{R/k}, M).$$

Proposition 31. *If $R = k[x_1,\ldots,x_r]$, then*

$$\Omega_{R/k} = \bigoplus_{i=1}^r R dx_i.$$

Proposition 32. *If $k \to R$ is a surjective map, then $\Omega_{R/k} = 0$.*

Proposition 33. *Given two ring homomorphisms $R \to S \to T$, we have the following right exact sequence of T-modules:*

$$\Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0.$$

If $T = S/I$ for some ideal I, then $\Omega_{T/S} = 0$ and we have the following right exact sequence (known as the conormal sequence):

$$I/I^2 \xrightarrow{[f]\mapsto df\otimes 1} \Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to 0.$$

Observe that the definition of the Kähler differentials $\Omega_{R/k}$ also makes sense when R is a dg algebra. In this case, the Kähler differentials become a dg module. Roughly speaking, the cotangent complex is the module resulting from this derived version of Kähler differentials.

5.2. **The cotangent complex.** A natural question to ask is whether the right exact sequences of the previous proposition can be extended to the left. In fact it can, and the answer is provided by Illusie's theory of the *cotangent complex* [Ill71], [Ill72] and the associated André-Quillen homology functors. The cotangent complex is a "homotopical" construction and is defined using simplicial machinery. However, the Dold-Kan correspondence provides an equivalence of categories between simplicial commutative rings and dg algebras over $\mathbb{Q}$. Thus we can use the machinery we have been developing so far to provide a direct construction of the cotangent complex.

For further details on the construction of the cotangent complex as a dg algebra, see [Man].

Definition 34. A dg k-algebra is *semifree* if the underlying graded algebra is a polynomial algebra over k.

Definition 35. A *k-semifree resolution* of a dg k-algebra A is a surjective quasi-isomorphism $R \to A$, where R is a semifree dg k-algebra.

If R is a dg k-algebra, then the Kähler differentials $\Omega_{R/k}$ become a dg R-module, where the grading is induced by the grading on R.

Definition 36. Let R and A be dg k-algebras, and let $R \to A$ be a k-semifree resolution. The *cotangent complex* $\mathbb{L}_{A/k}$ is the dg A-module defined as

$$\mathbb{L}_{A/k} := \Omega_{R/k} \otimes_R A.$$

Proposition 37. *If R is Noetherian and $A = R/I$, where I is generated by a regular sequence, then $\mathbb{L}_{B/A} \cong I/I^2[1]$ and I/I^2 is a projective module.*

Proof. [Qui70, Corollary 6.14] □

Proposition 38. *Let $A \to B \to C$ be a sequence of maps of dg k-algebras. We then have the following cofiber sequence or canonical distinguished triangle of dg C-modules:*

$$\mathbb{L}_{B/A} \otimes^L_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \to \mathbb{L}_{B/A} \otimes^L_B C[1].$$

Moreover, this cofiber sequence implies a long exact sequence in André-Quillen homology.

Proof. [Sta18, Tag 08QX]. □

As mentioned earlier, the cotangent complex can also be defined using simplicial methods by the commutative diagram of categories below, where $\simeq$ denotes Quillen-equivalence of categories.

$$\begin{array}{ccc} \mathbf{scRing}_k & \xrightarrow{\mathbb{L}} & \mathbf{sMod}_R \\ {\scriptstyle \simeq \text{ if } \mathbb{Q}\subseteq k \text{ by monoidal Dold-Kan}} \downarrow & & \downarrow {\scriptstyle \simeq \text{ by Dold-Kan}} \\ \mathbf{cdgAlg}_k & \xrightarrow[\mathbb{L}]{} & \mathbf{Ch}_R \end{array}$$

6. Derived intersections and symmetric algebras

In [ACH14], the authors proved that smooth derived intersections in local rings can be written as symmetric algebras and are formal over the corresponding smooth classical intersection. We now explore in what situations we can write a derived intersection in a quasi-smooth affine scheme as a symmetric algebra in a meaningful way, that is, when the derived intersection is $\operatorname{Spec}_X \operatorname{Sym}_X \mathcal{C}$ for some scheme X and some finite complex of locally free sheaves $\mathcal{C}$. As shown in the previous section, where we calculated various examples of quasi-smooth derived intersections, we notice that this is not always true in the quasi-smooth case. For instance, in Example 29 the derived intersection could not be written as a symmetric algebra in a meaningful way. However, Theorems 1 and 2 show two situations where this is possible; in this section we will prove these two theorems.

Throughout this section we inherit the notations from Section 4. In addition, we use the notation $R\{t_1, \dots, t_n\}$ as a shorthand for $Rt_1 \oplus \cdots \oplus Rt_n$.

Theorem 39. *Suppose $I \subset J$, then $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \operatorname{Sym}_{\overline{R}/\overline{J}} \mathbb{L}_{Y/k}$.*

Proof. Let $A = (a_1, \dots a_r)$ and $I = (f_1, \dots f_n)$ be regular ideals such that $A \subset I$. Write $a_j = \sum_{i=1}^n c_{ji} f_i$, where $c_{ji} \in R$. By Theorem 21, the polynomial dg algebra

$$\overline{R}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r],$$

where $|e_i| = -1$, $|\varepsilon_j| = -2$, $d(e_i) = f_i$ and $d(\varepsilon_j) = \sum_{i=1}^n c_{ji} e_i$ is a resolution of $\overline{R}/\overline{I}$. Thus, we obtain that

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r] \otimes_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}/\overline{J}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r].$$

Because $I \subset J$, we have that $f_1, \cdots, f_n \in J$. Therefore $d(e_i) = 0$ in $\overline{R}/\overline{J}$ for all i. We may therefore write $\overline{R}/\overline{J}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r]$ as

$$\cdots \to \overline{R}/\overline{J}\,\{\varepsilon_1, \dots, \varepsilon_r\} \oplus \bigoplus_{i<j} \overline{R}/\overline{J}\,\{e_i e_j\} \to \overline{R}/\overline{J}\,\{e_1, \dots, e_n\} \xrightarrow{0} \overline{R}/\overline{J}.$$

Hence we obtain that

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\overline{R}/\overline{J}} \left\{ \overline{R}/\overline{J}\varepsilon_1 \oplus \cdots \oplus \overline{R}/\overline{J}\varepsilon_r \xrightarrow{T} \overline{R}/\overline{J}e_1 \oplus \cdots \oplus \overline{R}/\overline{J}e_n \right\},$$

where the map T is the matrix (c_{ji}). On the other hand, we have that the cotangent complex $\mathbb{L}_{Y/k}$ is the dg $\overline{R}/\overline{J}$-module

$$\overline{R}/\overline{J}\,\{d\varepsilon_1, \dots, d\varepsilon_r\} \to \overline{R}/\overline{J}\,\{de_1, \dots, de_n\},$$

where $d(d\varepsilon_j) = \sum_{i=1}^n c_{ji} de_i$. Therefore $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\overline{R}/\overline{J}} \mathbb{L}_{Y/k}$. □

Proposition 40. *The map $\overline{R}/\overline{J}\,\{e_1, \dots, e_n\} \xrightarrow{d} \overline{R}/\overline{J}$ is zero if and only if $I \subset J$.*

Proof. The map d is zero if and only if $d(e_i) = 0$ in $\overline{R}/\overline{J}$ for all i, which precisely means $f_i \in J$ for i. Since $I = (f_1, \dots, f_n)$, this is same as saying $I \subset J$. □

Theorem 41. *Suppose $A \subset IJ$. Let $A = (a_1, \dots a_r)$ and $I = (f_1, \dots, f_n)$, then $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\widetilde{R}} \widetilde{R}^r[2]$, where $\widetilde{R}$ is the dg algebra $\overline{R}/\overline{J}[e_1, \dots, e_n]$ with $|e_i| = -1$ and with differential $d(e_i) = f_i$.*

Proof. Under the same setting of Theorem 39, we have

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r] \otimes_{\overline{R}} \overline{R}/\overline{J} \cong \overline{R}/\overline{J}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r].$$

where $|e_i| = -1$, $|\varepsilon_j| = -2$, $d(e_i) = f_i$ and $d(\varepsilon_j) = \sum_{i=1}^n c_{ji} e_i$. Since $A \subset IJ$, we are able to choose c_{ji} such that $c_{ji} \in J$, so that $d(\varepsilon_j) = 0$ in $\overline{R}/\overline{J}$ for all j. We may therefore write $\overline{R}/\overline{J}[e_1, \dots e_n, \varepsilon_1, \dots \varepsilon_r]$ as

$$\cdots \to \overline{R}/\overline{J}\,\{\varepsilon_1, \dots, \varepsilon_r\} \oplus \bigoplus_{i<j} \overline{R}/\overline{J} e_i e_j \xrightarrow{0} \overline{R}/\overline{J}\,\{e_1, \dots, e_n\} \to \overline{R}/\overline{J}.$$

If we ignore the ε_i's in the above complex, we obtain a copy of $\widetilde{R}$. Since $d(\varepsilon_i) = 0$, we can write the above complex as $\mathrm{Sym}_{\widetilde{R}} \widetilde{R}[\eta_1, \dots, \eta_r]$, where $|\eta_i| = -2$ and the η_i's represent the ε_i's (there is no differential on the η_i's). Therefore $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\widetilde{R}} \widetilde{R}^r[2]$. □

Proposition 42. *The map $\overline{R}/\overline{J}\,\{\varepsilon_1, \dots, \varepsilon_r\} \xrightarrow{d} \overline{R}/\overline{J}\,\{e_1, \dots, e_n\}$ is zero if and only if $A \subset IJ$.*

Proof. The "if" part is proven in Theorem 41. Suppose $d(\varepsilon_j) = 0$ in $\overline{R}/\overline{J}$ for all j, then $\sum_{i=1}^n c_{ji} e_i = 0$ in $\overline{R}/\overline{J}$ for all j. But $e_1, \dots, e_n$ are variables, so $c_{ji} = 0$ in $\overline{R}/\overline{J}$ for all i and j, which means $a_j = \sum_{i=1}^n c_{ji} f_i$ where $f_i \in I$ and $c_{ji} \in J$ for all i and j. Since $A = (a_1, \cdots, a_r)$, we obtain that $A \subset IJ$. □

7. Derived intersections and cotangent complexes

In this section we study the relationship between derived intersections in quasi-smooth affine schemes and cotangent complexes. We still use the same notations as in Section 4, and work in the quasi-smooth affine case. One may ask: can we always find the appropriate W' and E in different situations (if possible) such that the following two statements are equivalent?

(1) $X \times^L_S Y \cong \mathrm{Sym}_{W'} E$.
(2) $\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W \to \mathbb{L}_{Y/W}$ is a surjective splitting, that is, $\mathbb{L}_{Y/W}$ is isomorphic to a direct summand of $\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W$.

Here W' is a non-trivial cochain complex, meaning W' contains at least one term but is not the whole complex. In other words, condition (1) requires the derived intersection to be a symmetric algebra in a meaningful way.

The smooth version of this problem is studied in [ACH14], in which case one can just let $W' = W$ and both statements are always true. However, there are more obstructions in the quasi-smooth case. For example, in order to have a systematic way to calculate the cotangent complex $\mathbb{L}_{Y/W}$, we would like to use Tate resolution to resolve $\overline{R}/\overline{I+J}$ as an $\overline{R}/\overline{J}$-module as in Theorem 21. However, this requires $\overline{I}$ to be an ideal generated by a regular sequence in $\overline{R}/\overline{J}$, or equivalently, we need I to be an ideal generated by a regular sequence in R/J. This is not always true, but we will focus on the cases where this is true in order to use Tate resolutions.

First we state two theorems:

Theorem 43. *If $I \subset J$, then both statements are always true.*

Proof. Statement (1) is true by Theorem 39, and statement (2) is true because $Y = W$, so $\mathbb{L}_{Y/W} = 0$, and thus the map of cotangent complexes always splits. ☐

Theorem 44. *Suppose $A \subset IJ$ and the image of I in R/J is generated by a regular sequence, then both statements are always true.*

Proof. Statement (1) is true by Theorem 41. To prove statement (2), first notice that

$$\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W = \begin{array}{c} \overline{R}/\overline{I+J}\{d\varepsilon_1, \dots, d\varepsilon_r\} \\ \big\downarrow 0 \\ \overline{R}/\overline{I+J}\{de_1, \dots, de_n\} \end{array},$$

where the map is zero by Proposition 42. Also, since the image of I in R/J is generated by a regular sequence, the image of $\overline{I}$ in $\overline{R}/\overline{J}$ is generated by a regular sequence. Let $R' = \overline{R}/\overline{J}$, then $R' \cong R/J$, which a regular local ring by assumption. Then we can use Koszul resolution to resolve $R'/\overline{I}$ as an R'-module. Because Spec $R' = Y$ and Spec $R'/\overline{I} = W$, this allows us to compute $\mathbb{L}_{Y/W}$, which contains only degree one term, namely $\mathbb{L}_{Y/W} = \overline{R}/\overline{I+J}\{de'_1, \dots, de'_m\}$, where $m \leq n$ since $\overline{I}$ cannot have more generators in $\overline{R}/\overline{J}$ than in $\overline{R}$. Therefore $\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W \to \mathbb{L}_{Y/W}$ is

$$\begin{array}{ccc} \overline{R}/\overline{I+J}\{d\varepsilon_1, \dots, d\varepsilon_r\} & \longrightarrow & 0 \\ \big\downarrow 0 & & \big\downarrow \\ \overline{R}/\overline{I+J}\{de_1, \dots, de_n\} & \xrightarrow{\pi} & \overline{R}/\overline{I+J}\{de'_1, \dots, de'_m\} \end{array}$$

where the map π is given by the image of generators of $\overline{I}$ in $\overline{R}/\overline{J}$. This is a surjective splitting, so statement (2) is also true. ☐

The following example illustrates Theorem 44:

Example 45. Let $R = k[x, y]$, $A = (xy)$, $I = (x)$, and $J = (y)$. By Example 27,

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{k[x]}(k[x] \xrightarrow{\cdot x} k[x]),$$

so we can take $W' = \overline{R}/\overline{J} \cong k[x]$ and (1) is satisfied. After resolving $\overline{R}/\overline{I}$ using the Tate resolution, we have that $d(e) = x = 0$ and $d(\varepsilon) = ye$. Therefore

$$\mathbb{L}_{X/S} = \begin{array}{c} \overline{R}/\overline{I}\{d\varepsilon\} \\ \big\downarrow yde \\ \overline{R}/\overline{I}\{de\} \end{array}.$$

Note that $ye = 0$ after pulling back by $\overline{R}/\overline{J}$, so we obtain that

$$\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W = \begin{array}{c} \overline{R}/\overline{I+J}\{d\varepsilon\} \\ \big\downarrow 0 \\ \overline{R}/\overline{I+J}\{de\} \end{array} .$$

Since $W \cong k$ and $Y \cong k[x]$, we can use Koszul resolution to compute $\mathbb{L}_{Y/W}$, which contains only degree one term, namely $\mathbb{L}_{Y/W} = \overline{R}/\overline{I+J}\{de'\}$. Therefore $\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W \to \mathbb{L}_{Y/W}$ is

$$\begin{array}{ccc} \overline{R}/\overline{I+J}\{d\varepsilon\} & \longrightarrow & 0 \\ \big\downarrow 0 & & \big\downarrow \\ \overline{R}/\overline{I+J}\{de\} & \xrightarrow{\text{id}} & \overline{R}/\overline{I+J}\{de'\} \end{array}$$

which is a surjective splitting. Hence (2) is also satisfied.

However, besides the cases which fall into the above two theorems, the relationship between (1) and (2) is not clear in general. In the below example, both (1) and (2) are not easy to determine.

Example 46. Let $R = k[x, y, z]$, $A = (xyz)$, $I = (xy)$, and $J = (xz)$. By Example 29, there is no obvious way to write $\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J}$ as a non-trivial symmetric algebra. However, since I is not an ideal generated by a regular sequence in R/J, we couldn't use Tate resolution, so $\mathbb{L}_{Y/W}$ is also hard to calculate.

Moreover, observe that statement (1) is symmetric with respect to X and Y, but statement (2) is not. This leads us to the following counterexample, which shows that (1) and (2) are not equivalent.

Example 47. $R = k[x, y]$, $A = (xy^2)$, $I = (x)$, $J = (xy)$. Since $J \subset I$, by Theorem 39,

$$\overline{R}/\overline{I} \otimes^L_{\overline{R}} \overline{R}/\overline{J} \cong \mathrm{Sym}_{\overline{R}/\overline{I}}\, \mathbb{L}_{X/k},$$

that is, condition (1) is satisfied. We compute $\mathbb{L}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_W \to \mathbb{L}_{Y/W}$ to be

$$\begin{array}{ccc} \overline{R}/\overline{I+J}\{d\varepsilon\} & \xrightarrow{y} & \overline{R}/\overline{I+J}\{d\varepsilon'\} \\ \big\downarrow y^2 & & \big\downarrow y \\ \overline{R}/\overline{I+J}\{de\} & \xrightarrow{\text{id}} & \overline{R}/\overline{I+J}\{de'\} \end{array}$$

which does not split. On the other hand, we can compute $\mathbb{L}_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_W \to \mathbb{L}_{X/W}$ to be

$$\begin{array}{ccc} \overline{R}/\overline{I+J}\{d\varepsilon\} & \longrightarrow & 0 \\ \big\downarrow y^2 & & \big\downarrow \\ \overline{R}/\overline{I+J}\{de\} & \longrightarrow & 0 \end{array}$$

which splits (observe that $\mathbb{L}_{X/W} = 0$ because $X = W$). Note that this direction satisfies Theorem 43.

Roughly speaking, we have that the cotangent complex generally does not capture all the information about the map $\overline{R}/\overline{J}\,\{e_i\} \to \overline{R}/\overline{J}$.

8. Cohomology of derived self-intersections

Taking the cohomology of a dg algebra loses important information (like the algebra structure), but nonetheless it is a useful invariant. The cohomology is well-defined, while the derived intersection is only defined up to quasi-isomorphism.

What follows are some fairly computational results on the cohomology of derived intersections, in the case that A is a principal ideal.

Theorem 48. *Let $S = \overline{R}/\overline{I}$ for convenience. Then if $A = (f_0)$, $I = (f_1, \dots f_n)$, and $f_0 = \sum c_i f_i$, and $(c_1, \dots c_n) = 1$, $H^i(S \otimes^L_{\overline{R}} S) \cong S^k$, where $k = \binom{n-1}{i}$.*

Proof. As proved in Theorem 39, the derived-self intersection is the symmetric algebra of the cotangent complex $B \to C$, where $B \cong S$ and $C \cong S^n$. Each term in the sequence is thus a direct sum of some $\mathrm{Sym}^i B \otimes \wedge^j C$, but $\mathrm{Sym}^i S \cong S$ so they are just $\wedge^j S$. In fact, the complex breaks down as a direct sum of complexes as follows:

$$\wedge^0 C \longrightarrow \wedge^1 C$$

$$\wedge^0 C \longrightarrow \wedge^1 C \longrightarrow \wedge^2 C$$

$$\wedge^0 C \longrightarrow \wedge^1 C \longrightarrow \wedge^2 C \longrightarrow \wedge^3 C$$

$$\vdots$$

$$\wedge^0 C \longrightarrow \wedge^1 C \longrightarrow \wedge^2 C \longrightarrow \cdots \longrightarrow \wedge^n C$$

As we can see, what we have are (truncated) copies of the complex $\wedge^0 C \to \cdots$, shifted by two degrees to the left, and each containing on more term than the one below it. Since $\wedge^{n+1} C = 0$, at some point the complexes all become $\wedge^0 C \to \cdots \to \wedge^n C$.

The differential is $a \mapsto a \wedge (c_1 e_1 \dots c_1 e_n)$, where $e_1, \dots, e_n$ is the basis of C. So it is the dual Koszul complex $K^*(c_1, \dots c_n)$! (cf [Eis95, Ch.17]) According to a theorem in [Ser00], Koszul (co)homology is annihilated by $(c_1, \dots c_n)$. Since we have assumed $(c_1, \dots, c_n) = 1$, the homology must all be zero; each complex $\wedge^0 C \to \cdots \wedge^n C$ is exact. Thus we are left with only the complexes $\wedge^0 C \to \wedge^1 C \to \cdots \to \wedge^i C$ for $i < n$. For these complexes, the only homology can be at the last term. It suffices to find the cokernels $\wedge^i C/(\ker(\wedge^i \to \wedge^{i+1})$. It is easy to see that this is isomorphic to $\wedge^i C'$, where $C' \cong S^{n-1}$. Since $\wedge^i(S^{n-1}) \cong S^k$ where $k = \binom{n-1}{i}$, we are done. ☐

Theorem 49. *Let $S = \overline{R}/\overline{I}$ for convenience. Then assume $A = (f_0)$, $I = (f_1, \dots f_n)$, and $f_0 = \sum c_i f_i$, and (the classes of) $c_1, \dots c_n$ are a regular sequence (in S). Then the derived intersection is*

$$H^i(S \otimes^L_{\overline{R}} S) \cong S^k \oplus \begin{cases} S/(c_1, \dots c_n) & \text{if } i \geq n \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

where $k = \binom{n-1}{i}$.

Proof. Let's first focus on the "full length" complexes $\wedge^0 C \to \cdots \to \wedge^n C$. Since $c_1, \dots, c_n$ is a regular sequence, Koszul homology vanishes everywhere except at the end (at $\wedge^n C$), where it is $S/(c_1, \dots, c_n)$, by Theorem 18. Thus we have many copies of $S/(c_1, \dots, c_n)$, spaced out by two. They end where we have the final occurrence of $\wedge^n$.

For the shorter complexes, we end up getting the same terms of $\wedge^i C'$, via a direct computation. These form the contribution of S^k to the cohomology of the derived intersection. ☐

9. A commutative algebra theorem

In this section we state and prove a commutative algebra theorem which helps us understand the assumptions of the theorems in the previous section better.

Theorem 50. *Let $A = (f_0)$, $I = (f_1, \dots f_n)$ where $\{f_1, \dots, f_n\}$ is a minimal generating set of I. Assume I is the maximal ideal of R , then there exists some $c_1, \dots, c_n \in R$ such that $f_0 = \sum c_i f_i$ and that $c_1, \dots, c_n$ satisfy one of the three following cases:*

(1) $c_i = 0$ for some i.
(2) c_i is a unit for some i.
(3) $c_1, \dots, c_n$ is a regular sequence.

Furthermore, if f_0 is not contained in any ideal generated by a proper sub-sequence of $f_1, \dots, f_n$, then we may let $c_1, \dots, c_n$ satisfy either case (2) or case (3).

Remark 51. Cases (2) and (3) correspond to Theorems 48 and 49, respectively.

Lemma 52. *Each f_i is prime in R.*

Proof of Lemma 52. Note that any regular local ring is a UFD, so we just need to prove each f_i is irreducible in R. Suppose not, without loss of generality assume f_1 is not irreducible, let $f_1 = f_1' f_1''$ where f_1', f_1'' are not units. Then $(f_1', f_2, \dots, f_n)$ doesn't contain units and $I \subset (f_1', f_2, \dots, f_n)$. Therefore $I = (f_1', f_2, \dots, f_n)$. So $\{f_1', f_2, \dots, f_n\}$ is also a minimal generating set of I. Because I is generated by some regular sequence and R is Cohen-Macaulay, we have that $f_1', f_2, \dots, f_n$ is a regular sequence. Write $f_1' = f_1 r_1 + \cdots + f_n r_n$ where $r_i \in R$. Because $f_1 \in (f_1')$, $f_2 r_2 + \cdots + f_n r_n \in (f_1')$, so by regularity $r_2, \dots, r_n = 0$. Therefore $f_1' = f_1 r_1$, which is a contradiction. Thus each f_i is irreducible in R. □

Proof of Theorem 50. The case $n = 1$ is trivial so let $n \geq 2$. Write $f_0 = \sum d_i f_i$ for some $d_i \in R$. If $d_1, \dots, d_n$ is a regular sequence, let $c_i = d_i$ and then $c_1, \dots, c_n$ satisfy case (3). Also, if there exists some i where $d_i = 0$, let $c_i = d_i$ and then $c_1, \dots, c_n$ satisfy case (1).

Suppose $d_1, \dots, d_n$ is not a regular sequence and $d_1, \dots, d_n \neq 0$, then d_n is a zero-divisor in $R/(d_1, \dots, d_{n-1})$, so $d_n | d_1 \cdots d_{n-1}$. If $f_i \nmid d_n$ for each i, then $d_n \in R - I$. Since I is maximal, $d_n \in R - I$ implies d_n is a unit. Then let $c_i = d_i$ and $c_1, \dots, c_n$ satisfy case (2).

Now suppose $d_n \neq 0$ is also not a unit, then there exists f_j such that $f_j | d_n$. If $j = n$, then $d_n | d_1 \cdots d_{n-1}$ implies $f_n | d_1 \cdots d_{n-1}$. By Lemma 50, we have that f_n is prime, so $f_n | d_k$ for some $1 \leq k \leq n-1$, say $d_k = r f_n$. Then $f_0 = \sum d_i f_i = (\sum_{i=1, i \neq k,n}^n d_i f_i) + d_k f_k + d_n f_n = (\sum_{i=1, i\neq k,n}^n d_i f_i) + r f_n f_k + d_n f_n = (\sum_{i=1,i\neq k,n}^n d_i f_i) + f_n(r f_k + d_n)$. Let $c_i = d_i$ when $i \neq k, n$, $c_n = r f_k + d_n$, $c_k = 0$, then $f_0 = \sum c_i f_i$ and $c_1, \dots, c_n$ satisfy case (1). If $j \neq n$, then let $d_n = r f_j$. Similarly $f_0 = \sum d_i f_i = (\sum_{i=1,i\neq j,n}^n d_i f_i) + d_j f_j + d_n f_n = (\sum_{i=1,i\neq j,n}^n d_i f_i) + d_j f_j + r f_j f_n = (\sum_{i=1,i\neq k,n}^n d_i f_i) + f_j(d_j + r f_n)$. Let $c_i = d_i$ when $i \neq j, n$, $c_j = d_j + r f_n$, $c_n = 0$, then $f_0 = \sum c_i f_i$ and $c_1, \dots, c_n$ satisfy case (1). □

10. Further questions: derived fixed points

As a particular case of derived intersections, we may study the derived fixed points of an affine scheme under an automorphism.

Let X be an affine scheme over k, and let φ be an automorphism of X. Both the diagonal of X, denoted by Δ_X, and the graph of f, written as Γ_φ, are closed subschemes of the fiber product $X \times_k X$. The *derived fixed points of X under φ* are defined to be the derived fiber product

$$X^\varphi := \Delta_X \times^L_{X \times_k X} \Gamma_\varphi.$$

In other words, the derived fixed points are defined to be the derived version of the pullback diagram below.

$$\begin{array}{ccc} X^\varphi & \longrightarrow & \Gamma_\varphi \\ \big\downarrow & & \big\downarrow \\ \Delta_X & \longrightarrow & X \times_k X \end{array}$$

Assume that $X = \operatorname{Spec} \overline{R}/\overline{I}$ is a quasi-smooth affine scheme. We have that both Δ_X and Γ_φ are isomorphic to X, so $\Delta_X = \operatorname{Spec} \overline{R}/\overline{I}$ and $\Gamma_\varphi = \operatorname{Spec} \overline{R}/\overline{I}$.

From the algebraic point of view, we want to look at the derived tensor product corresponding to the pushout diagram

$$\begin{array}{ccc} \overline{R}/\overline{I} \otimes^L_{\overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I}} \overline{R}/\overline{I} & \longleftarrow & \overline{R}/\overline{I} \\ \big\uparrow & & \big\uparrow \Gamma \\ \overline{R}/\overline{I} & \xleftarrow{\quad \Delta \quad} & \overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I} \end{array}$$

where $\Delta : r \otimes s \mapsto rs$ and $\Gamma : r \otimes s \mapsto r\varphi(s)$.

To compute this derived tensor product, we may resolve the ring $\overline{R}/\overline{I}$ corresponding to the diagonal as an $\overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I}$-module. In this case, we have that

$$\overline{R}/\overline{I} \cong \left(\overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I} \right)/M,$$

where M is the kernel of the multiplication map $\overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I} \to \overline{R}/\overline{I}$. In particular, M is generated by all elements of the form $r \otimes 1 - 1 \otimes r$, where $r \in \overline{R}/\overline{I}$. Moreover, we can choose generators of M by using

Nakayama's lemma. Therefore, we may apply the Koszul-Tate resolution on $\left(\overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I}\right)/M$ to resolve $\overline{R}/\overline{I}$ as an $\overline{R}/\overline{I} \otimes_k \overline{R}/\overline{I}$-module.

Question 53. *Let X be quasi-smooth. When can X^{φ} be written as a symmetric algebra?*

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